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# INEGALITĂŢI TRIGONOMETRICE

## DE LA INIŢIERE LA PERFORMANŢĂ

ÎNVĂȚARE DE EXCELENȚĂ®
supersucces

# Cuprins

# Considerații preliminare

*„Un om de succes este acela care poate construi o fundație solidă cu cărămizile pe care alții le aruncă în el."*

David Charles Brink (n. 1939)

## 1. Notații

$A, B, C$ – măsurile unghiurilor unui triunghi.
$a, b, c$ – lungimile laturilor $BC$, $CA$, $AB$.
$h_a, h_b, h_c$ – lungimile înălțimilor.
$m_a, m_b, m_c$ – lungimile medianelor.
$l_a, l_b, l_c$ – lungimile bisectoarelor interioare.
$r_a, r_b, r_c$ – razele cercurilor exînscrise.
$O$ – centrul cercului circumscris.
$I$ – centrul cercului înscris.
$H$ – ortocentrul.
$G$ – centrul de greutate.
$r$ – raza cercului înscris.
$R$ – raza cercului circumscris.
$S$ – aria triunghiului.
$p$ – semiperimetrul.

## 2. Egalități (identități) remarcabile în triunghi

$$IG^2 = \frac{1}{9}\left(p^2 + 5r^2 - 16Rr\right); \quad IH^2 = 4R^2 + 4Rr + 3r^2 - p^2; \quad IO^2 = R^2 - 2Rr;$$

$$OH^2 = 9R^2 + 8Rr + 2r^2 - 2p^2; \quad OG^2 = R^2 - \frac{1}{9}\left(a^2 + b^2 + c^2\right);$$

$$\sum bc = p^2 + r^2 + 4Rr; \quad \sum a^2 = 2\left(p^2 - r^2 - 4Rr\right); \quad \sum a^3 = 2p\left(p^2 - 3r^2 - 6Rr\right);$$

$$\sum a^4 = 2\left[p^4 - 2p^2r(4R+3r) + r^2(4R+r)^2\right]; \quad \prod(b+c) = 2p\left(p^2 + r^2 + 2Rr\right);$$

$$\sum b^2c^2 = p^4 - 2p^2r(4R-r) + r^2(4R+r)^2; \quad \sum \frac{a}{b+c} = \frac{2\left(p^2 - r^2 - Rr\right)}{p^2 + r^2 + 2Rr} \ge \frac{3}{2};$$

$$\sum \frac{(b+c)^2}{bc} = \frac{p^2 + r^2 + 10Rr}{2Rr} \ge 12; \quad \sum \frac{1}{p-a} = \frac{4R+r}{rp} \ge \frac{\sqrt{3}}{r};$$

$$\sum \frac{a}{p-a} = \frac{2(2R-r)}{r} \ge 6; \quad \sum \frac{a^2}{p-a} = \frac{4p(R-r)}{r} \ge 4p; \quad \sum \frac{p-a}{a} = \frac{p^2 + r^2 - 8Rr}{4Rr} \ge \frac{3}{2};$$

$$\sum \frac{1}{a(p-a)} = \frac{p^2 + (4R+r)^2}{4Rrp^2} \ge \frac{1}{Rr}; \quad \sum (p-b)(p-c) = r(4R+r);$$

$$\sum \frac{bc}{p-a} = \frac{p^2 + (4R+r)^2}{p} \ge 4p; \quad \sum \frac{bc}{(p-b)(p-c)} = \frac{p^2 + r^2 - 8Rr}{r^2} \ge 12;$$

$$\sum bc(p-b)(p-c)=r^2\left[p^2+(4R+r)^2\right]\geq 4r^2p^2;\quad \sum(p-a)^2=p^2-2r(4R+r);$$

$$\sum\frac{1}{bc(p-b)(p-c)}=\frac{4R+r}{2Rr^2p^2};\quad \sum\frac{1}{b+c}=\frac{5p^2+r^2+4Rr}{2p\left(p^2+r^2+2Rr\right)};$$

$$\sum\frac{a^2}{bc}=\frac{p^2-3r^2-6Rr}{2Rr}\geq 3;\quad \sum\frac{(b+c)^2}{b+c-a}=\frac{2p(R+2r)}{r}\geq 8p;$$

$$\sum\frac{1}{(p-a)^2}=\frac{(4R+r)^2-2p^2}{r^2p^2}\geq\frac{1}{r^2};\quad \sum\frac{bc}{a(b+c)}\leq\frac{p^2+r^2-2Rr}{8Rr};$$

$$\sum\frac{b+c}{a}=\frac{p^2+r^2-2Rr}{2Rr}\geq 6;\quad \sum\frac{1}{(p-b)(p-c)}=\frac{1}{r^2};\quad 2\sum bc-\sum a^2=4r(4R-r);$$

$$\sum bc-\sum(p-a)^2=3r(4R+r);\quad \sum a(p-a)^2=2rp(2R-r);$$

$$\sum a^2(p-a)=4rp(R+r);\quad \sum bc(p-a)=p\left(p^2+r^2-8Rr\right);$$

$$\sum bc(p-a)^2=p^2\left(p^2+r^2-12Rr\right);\quad \sum\frac{b^2+c^2-a^2}{b+c}=\frac{p^4+8r^2p^2-r^2(4R+r)^2}{p\left(p^2+r^2+2Rr\right)};$$

$$\sum a(p-b)(p-c)=2rp(2R-r);\quad \sum a^2(p-b)(p-c)=4rp^2(R-r);$$

$$2\sum b^2c^2-\sum a^4=16S^2;\quad \sum\frac{a^2}{(p-b)(p-c)}=\frac{4(R+r)}{r}\geq 12;\quad \sum\frac{a(b+c)}{p(p-a)}=\frac{4R}{r}\geq 8;$$

$$\sum\frac{bc}{p(p-a)}=1+\left(\frac{4R+r}{p}\right)^2\geq 4;\quad \sum\frac{p(p-a)}{bc}=\frac{4R+r}{2R}\leq\frac{9}{4};\quad \sum\frac{(p-a)^2}{bc}=1-\frac{r}{2R};$$

$$\sum\frac{1}{a^2}=\frac{p^4-2p^2r(4R-r)+r^2(4R+r)^2}{16R^2r^2p^2}\leq\frac{1}{4r^2};\quad \sum r_a=4R+r;\quad \sum r_br_c=p^2;$$

$$\prod r_a=rp^2;\quad \sum h_a=\frac{p^2+r^2+4Rr}{2R};\quad \sum h_bh_c=\frac{2rp^2}{R};\quad \prod h_a=\frac{2r^2p^2}{R};$$

$$\sum ar_a=2p(2R-r);\quad \sum a^2r_a=4p^2(R-r);\quad \sum r_a^3=(4R+r)^3-12Rp^2;$$

$$\sum a\cdot r_br_c=2rp(4R+r);\quad \sum\frac{a}{r_br_c}=\frac{2(2R-r)}{rp};\quad \sum bc\cdot r_br_c=p^2\left(p^2+r^2-8Rr\right);$$

$$\sum r_a^2=(4R+r)^2-2p^2;\quad \sum\frac{h_b+h_c}{h_a}=\frac{p^2+r^2-2Rr}{2Rr}\geq 6;\quad \sum\frac{h_bh_c}{ah_a}=\frac{p}{R}\leq\frac{3\sqrt{3}}{2};$$

$$\sum\frac{h_b+h_c}{r_a}=6;\quad \sum\frac{r_b+r_c}{r_a}=\frac{2(2R-r)}{r}\geq 6;\quad \sum\frac{r_b+r_c}{h_a}=\frac{2(R+r)}{r}\geq 6;$$

$$\sum\frac{h_b+h_c}{r_b+r_c}=\frac{2(R+r)}{R}\leq 3;\quad \sum\frac{a^2}{r_br_c}=\frac{4(R-r)}{r}\geq 4;\quad \sum\frac{bc}{r_br_c}=1+\frac{(4R+r)^2}{p^2};$$

$$\sum\frac{h_bh_c}{a^2}=\frac{p^2+r^2+4Rr}{4R^2}\leq\frac{9}{4};\quad \sum\frac{r_b+r_c}{a}=\frac{p}{r}\geq 3\sqrt{3};\quad \sum\frac{h_b+h_c}{a}=\frac{2p}{R}\leq 3\sqrt{3};$$

$$\sum\frac{h_a^2}{bc}=\frac{p^2+r^2+4Rr}{4R^2}\le\frac{9}{4};\quad \sum\frac{r_br_c}{bc}=\frac{4R+r}{2R}\le\frac{9}{4};\quad \sum\frac{h_bh_c}{bc}=\frac{p^2-r^2-4Rr}{2R^2}\le\frac{9}{4};$$

$$\sum\frac{r_a}{a}=\frac{p^2+(4R+r)^2}{4Rp};\quad \sum\frac{a}{r_a}=\frac{2(4R+r)}{p}\ge2\sqrt{3};\quad \sum\frac{a^2}{r_a}=4(R+r);$$

$$\sum\frac{a}{r_a^2}=\frac{2(2R-r)}{rp};\quad \sum\frac{a^2}{r_a^2}=\frac{2}{p^2}\left[(4R+r)^2-p^2\right]\ge4;\quad \sum\frac{1}{r_a^2}=\frac{p^2-2r(4R+r)}{r^2p^2};$$

$$\sum\frac{r_a^2}{a^2}=\frac{(4R+r)\left[(4R+r)^3-2p^2(4R-r)\right]+p^4}{16R^2p^2};\quad \sum\frac{r_a}{bc}=\frac{2R-r}{2Rr};$$

$$\sum\frac{a^3}{r_a}=\frac{2p^2(2R+3r)-2r(4R+r)^2}{p};\quad \sum\frac{r_br_c}{a}=\frac{p\left(p^2+r^2-8Rr\right)}{4Rr};\quad \sum\frac{h_bh_c}{a}=\frac{3rp}{R};$$

$$\sum\frac{r_a}{r_b+r_c}=\frac{(4R+r)^3+p^2(r-8R)}{4Rp^2};\quad \prod(r_b+r_c)=4Rp^2;\quad \sum\frac{a(b+c)}{r_a}=12R;$$

$$\sum bc\cdot r_a=r\left[p^2+(4R+r)^2\right];\quad \sum\frac{r_a-r}{r_a+r}=\sum\frac{a}{b+c}=\frac{2\left(p^2-r^2-Rr\right)}{p^2+r^2+2Rr};$$

$$\sum\frac{h_a-r}{h_a+r}=\sum\frac{2p-a}{2p+a}=\frac{15p^2-10Rr-r^2}{9p^2+6Rr+r^2};\quad \frac{h_a-2r}{h_a+2r}=\frac{p-a}{p+a};\quad \sum\frac{2R-r_a}{bc}=\frac{1}{2R}\le\frac{1}{4r};$$

$$\sum\frac{h_b+h_c}{r_b+r_c}=\frac{2(R+r)}{R}\le3;\quad \sum\frac{1}{r_b+r_c}=\frac{p^2+(4R+r)^2}{4Rp^2};\quad \sum\frac{a}{h_a}=\frac{p^2-r^2-4Rr}{rp};$$

$$\sum\frac{1}{h_ar_a}=\frac{4R+r}{rp^2};\quad \sum h_ar_a=\frac{r}{2R}\left[p^2+(4R+r)^2\right];\quad \sum\frac{bc}{ah_a}=\frac{p^2+r^2+4Rr}{2rp}\ge2\sqrt{3};$$

$$\sum\frac{h_a}{a}=\frac{p^4-2p^2r(4R-r)+r^2(4R+r)^2}{8R^2rp};\quad \sum\frac{bc}{h_bh_c}=\frac{p^4-2p^2r(4R-r)+r^2(4R+r)^2}{4r^2p^2};$$

$$\sum\frac{a(b+c)}{h_bh_c}=\frac{4R}{r}\ge8;\quad \sum\frac{a(b+c)}{r_br_c}=\frac{4R}{r};\quad \sum\frac{bc}{h_a^2}=\frac{2R}{r}\ge4;\quad \sum\frac{bc}{r_a^2}=\frac{p^2+r^2-12Rr}{r^2}\ge4;$$

$$\sum\frac{bc}{h_ar_a}=\frac{2R}{r}\ge4;\quad \sum\frac{ar_a}{bc}=\frac{p(R-r)}{Rr};\quad \sum\frac{r_br_c}{ar_a}=\frac{p\left(p^2+r^2-12Rr\right)}{4Rr^2};\quad \sum\frac{p(p-a)}{r_br_c}=3;$$

$$\sum\frac{ah_a}{bc}=\frac{p}{R}\le\frac{3\sqrt{3}}{2};\quad \sum\frac{r_br_c}{ah_a}=\frac{p}{2r}\ge\frac{3\sqrt{3}}{2};\quad \sum\frac{p(p-a)}{h_bh_c}=\frac{p^2+r^2-8Rr}{4r^2}\ge3;$$

$$\sum\frac{h_bh_c}{ar_a}=\frac{p}{R}\le\frac{3\sqrt{3}}{2};\quad \sum\frac{p(p-a)}{h_ar_a}=\frac{2R}{r}-1\ge3;\quad \sum\frac{(p-b)(p-c)}{ar_a}=\frac{p^2+r^2+4Rr}{4Rp};$$

$$\sum\frac{(p-b)(p-c)}{h_ar_a}=1;\quad \sum\frac{(p-b)(p-c)}{r_a^2}=1;\quad \sum\frac{(p-b)(p-c)}{ah_a}=\frac{4R+r}{2p}\ge\frac{\sqrt{3}}{2};$$

$$\sum\frac{ar_a}{(p-b)(p-c)}=\frac{2p}{r}\ge6\sqrt{3};\quad \prod\left(h_a-2r\right)=\frac{2r^4}{R};\quad \sum\frac{ah_a}{(p-b)(p-c)}=\frac{2p}{r}\ge6\sqrt{3};$$

$$\sum\frac{r_a^2}{(p-b)(p-c)}=\frac{4R+r}{r};\quad \sum\frac{bc}{r_br_c}=\left(\frac{4R+r}{p}\right)^2\ge 3;\quad \sum\frac{b+c}{h_a}=\frac{p^2+r^2+4Rr}{rp};$$

$$\sum\frac{a^2}{h_a}=\frac{p^2-3r^2-6Rr}{r};\quad \sum\frac{a}{h_a^2}=\frac{p^2-3r^2-6Rr}{2r^2p};\quad \sum\frac{a}{h_b+h_c}=\frac{R}{p}\cdot\frac{5p^2+r^2+4Rr}{p^2+r^2+2Rr}\ge\sqrt{3};$$

$$\sum\frac{a(b+c)}{h_a}=\frac{p^2+r^2-2Rr}{r};\quad \sum\frac{h_a}{l_a^2}=\frac{R+2r}{2Rr};\quad \sum\frac{h_bh_c}{h_b+h_c}=r\cdot\frac{5p^2+r^2+4Rr}{p^2+r^2+2Rr};$$

$$\sum\frac{1}{h_bh_c}=\frac{p^2+r^2+4Rr}{4r^2p^2};\quad \sum\frac{1}{l_a^2}=\frac{p^2(2R+r)+r^2(4R+r)}{8Rr^2p^2};\quad \prod l_a=\frac{16Rr^2p^2}{p^2+r^2+2Rr};$$

$$\sum\sin A=4\prod\cos\frac{A}{2}=\frac{p}{R}\le\frac{3\sqrt{3}}{2};\quad \sum\sin B\sin C=\frac{p^2+r^2+4Rr}{4R^2}\le\frac{9}{4};$$

$$\prod\sin A=\frac{pr}{2R^2}=\frac{S}{2R^2}\le\frac{3\sqrt{3}}{8};\quad \sum\frac{1}{\sin A}=\frac{p^2+r^2+4Rr}{2rp}\ge 2\sqrt{3};$$

$$\sum\cos A=1+4\prod\sin\frac{A}{2}=1+\frac{r}{R}\le\frac{3}{2};\quad \sum\cos B\cos C=\frac{p^2+r^2-4R^2}{4R^2}\le\frac{3}{4};$$

$$\prod\cos A=\frac{p^2-(2R+r)^2}{4R^2}=\frac{a^2+b^2+c^2-8R^2}{8R^2}\le\frac{1}{8};\quad \sum\frac{1}{\cos A}=\frac{p^2+r^2-4R^2}{p^2-(2R+r)^2}\ge 6;$$

$$\sum\operatorname{tg}B\operatorname{tg}C=\frac{p^2-r^2-4Rr}{p^2-(2R+r)^2};\quad \sum\sin^2 A=2+2\prod\cos A=\frac{p^2-r^2-4Rr}{2R^2}\le\frac{9}{4};$$

$$\sum\cos^2 A=\frac{6R^2+4Rr+r^2-p^2}{2R^2}\ge\frac{3}{4};\quad \sum\cos^2 A+2\prod\cos A=1;\quad \sum\operatorname{ctg}B\operatorname{ctg}C=1;$$

$$\sum\frac{1}{\sin B\sin C}=\frac{2R}{r}\ge 4;\quad \sum\frac{1}{\cos B\cos C}=\frac{4R(R+r)}{p^2-(2R+r)^2};\quad \prod(1-\cos A)=\frac{r^2}{2R^2}\le\frac{1}{8};$$

$$\sum\cos(B-C)=\frac{p^2+r^2+2Rr}{2R^2}-1;\quad \sum\sin 2A=\frac{2rp}{R^2};\quad \prod\sin 2A=\frac{rp\left[p^2-(2R+r)^2\right]}{r^4};$$

$$\sum\sin 2A=4\prod\sin A;\quad \sum\sin 4A=-4\prod\sin 2A;\quad \sum a\sin A=\frac{p^2-r^2-4Rr}{R};$$

$$\sum a\cos A=2a\sin B\sin C=\frac{2rp}{R}\le p;\quad \sum a\cos B\cos C=2R\prod\sin A=\frac{pr}{R}\le\frac{p}{2};$$

$$\sum a^2\cos A=\frac{r\left[3p^2-(2R+r)(4R+r)\right]}{p};\quad \sum a\cos^2 A=\frac{p\left(4R^2+6Rr+3r^2-p^2\right)}{2R^2};$$

$$\sum a^3\cos(B-C)=3abc;\quad \sum\frac{a}{\cos A}=\frac{4SR}{p^2-(2R+r)^2};\quad \sum a\sin B\sin C=\frac{3pr}{R};$$

$$\sum\sin^3 A=\frac{2p\left(p^2-6Rr-3r^2\right)}{8R^3};\quad \sum\cos^3 A=\frac{(2R+r)^2-3rp^2}{4R^3};$$

$$\sum \sin^4 A = \frac{p^4 - 2p^2 r(R+3r) + r^2(4R+r)^2}{8R^4};\ \sum \operatorname{tg} A = \prod \operatorname{tg} A = \frac{2rp}{p^2 - (2R+r)^2};$$

$$\sum \operatorname{tg} A = \prod \operatorname{tg} A \Leftrightarrow \sum \operatorname{ctg} B \operatorname{ctg} C = 1;\ \sum \operatorname{ctg} A = \frac{a^2+b^2+c^2}{4S} = \frac{p^2 - r^2 - 4Rr}{2rp} \geq 3;$$

$$\prod \operatorname{ctg} A = \frac{p^2 - (2R+r)^2}{2rp};\ \prod(\operatorname{tg} A + \operatorname{tg} B) = \frac{8R^2 rp}{\left[p^2 - (2R+r)^2\right]^2};$$

$$\prod(\operatorname{ctg} A + \operatorname{ctg} B) = \frac{2R^2}{rp};\quad \sin\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{bc}};\ \cos\frac{A}{2} = \sqrt{\frac{p(p-a)}{bc}};$$

$$\operatorname{tg}\frac{A}{2} = \sqrt{\frac{(p-b)(p-c)}{p(p-a)}};\ \operatorname{ctg}\frac{A}{2} = \sqrt{\frac{p(p-a)}{(p-b)(p-c)}};\ \sin^2\frac{A}{2} = \frac{1-\cos A}{2};$$

$$\cos^2\frac{A}{2} = \frac{1+\cos A}{2};\ \prod \sin\frac{A}{2} = \frac{r}{4R} \leq \frac{1}{8};\ \prod \cos\frac{A}{2} = \frac{p}{4R} \leq \frac{3\sqrt{3}}{8};\ \sum \frac{1}{a}\sin^2\frac{A}{2} = \frac{4R+r}{4Rp};$$

$$\sum \sin^2\frac{A}{2} = 1 - \frac{r}{2R} \geq \frac{3}{4};\quad \sum \cos^2\frac{A}{2} = 2 + \frac{r}{2R} \leq \frac{9}{4};\quad \sum \operatorname{cosec}\frac{A}{2} = 4R\sum \cos\frac{A}{2} \geq 6$$

$$\sum \frac{1}{\sin^2\frac{A}{2}} = \frac{p^2 + r^2 - 8Rr}{r^2} \geq 12;\ \sum \frac{1}{\cos^2\frac{A}{2}} = \frac{p^2 + (4R+r)^2}{p^2} \geq 4;\ \prod \operatorname{tg}\frac{A}{2} = \frac{r}{p} \leq \frac{1}{3\sqrt{3}};$$

$$\sum \frac{\sin\frac{A}{2}}{\cos\frac{B}{2}\cos\frac{C}{2}} = 2;\ \sum \frac{\cos\frac{A}{2}}{\sin\frac{B}{2}\sin\frac{C}{2}} = \frac{2p}{r};\ \sum \operatorname{tg}\frac{A}{2} = \frac{4R+r}{p} \geq \sqrt{3};\ \sum \operatorname{tg}\frac{A}{2}\operatorname{tg}\frac{B}{2} = 1;$$

$$\sum \operatorname{ctg}\frac{A}{2} = \frac{p}{r} \geq 3\sqrt{3};\ \prod \operatorname{ctg}\frac{A}{2} = \frac{p}{r} \geq 3\sqrt{3};\ \sum \operatorname{ctg}\frac{A}{2}\operatorname{ctg}\frac{B}{2} = \frac{4R+r}{r} \geq 9;$$

$$\sum \operatorname{tg}^2\frac{A}{2} = \left(\frac{4R+r}{p}\right)^2 - 2 \geq 1;\ \sum \operatorname{ctg}^2\frac{A}{2} = \frac{p^2 - 2r^2 - 8Rr}{r^2} \geq 9;$$

$$\sum \cos^2\frac{B-C}{2} = \frac{p^2 + r^2 + 2Rr}{4R^2} + 1;\ \prod \cos\frac{B-C}{2} = \frac{p^2 + r^2 + 2Rr}{8R^2};$$

$$\sum \sec^2\frac{A}{2}\sec^2\frac{B}{2} = \frac{8R(4R+r)}{p^2} \leq \frac{4R^2}{3r^2};\ \sum \operatorname{cosec}^2\frac{A}{2}\operatorname{cosec}^2\frac{B}{2} = \frac{8R(2R-r)}{r^2} \geq 48;$$

$$\sum \sin^2\frac{B}{2}\sin^2\frac{C}{2} = \frac{p^2 + r^2 - 8Rr}{16R^2};\ \sum \cos^2\frac{B}{2}\cos^2\frac{C}{2} = \frac{p^2 + (4R+r)^2}{16R^2} \geq \frac{p^2}{4R^2};$$

$$\sum \operatorname{tg}^2\frac{B}{2}\operatorname{tg}^2\frac{C}{2} = \frac{p^2 - 2r^2 - 8Rr}{p^2} \geq \frac{1}{3};\ \sum \operatorname{ctg}^2\frac{B}{2}\operatorname{ctg}^2\frac{C}{2} = \frac{(4R+r)^2 - 2p^2}{r^2} \geq 27;$$

$$\sum \sin^4\frac{A}{2} = \frac{8R^2 + r^2 - p^2}{8R^2};\ \sum \cos^4\frac{A}{2} = \frac{(4R+r)^2 - p^2}{8R^2} \geq \frac{p^2}{4R^2};$$

$$\prod\left(\operatorname{tg}\frac{B}{2}+\operatorname{tg}\frac{C}{2}\right)=\frac{4R}{p};\quad \prod\left(\operatorname{ctg}\frac{B}{2}+\operatorname{ctg}\frac{C}{2}\right)=\frac{4Rp}{r^2};\quad \sum\operatorname{tg}^3\frac{A}{2}=\frac{p\left(p^2-12Rr\right)}{r^3};$$

$$\sum a\cdot\sin^2\frac{A}{2}=\frac{p(R-r)}{R};\quad \sum a\cdot\cos^2\frac{A}{2}=\frac{p(R+r)}{R};\quad \sum bc\sin^2\frac{A}{2}=r(4R+r);$$

$$\sum bc\cos^2\frac{A}{2}=p^2;\quad \sum\frac{1}{a}\cdot\sin^2\frac{A}{2}=\frac{4R+r}{4Rp};\quad \sum\frac{1}{a}\cdot\cos^2\frac{A}{2}=\frac{p}{4Rr};$$

$$\sum a\operatorname{tg}\frac{A}{2}=2(2R-r);\quad \sum a\operatorname{ctg}\frac{A}{2}=2(4R+r)\geq 18r;\quad \sum\frac{\sin^2\frac{A}{2}}{bc}=\frac{R-r}{4R^2r};$$

$$\sum\frac{\cos^2\frac{A}{2}}{bc}=\frac{R+r}{4R^2r};\quad \sum\frac{\operatorname{tg}^2\frac{A}{2}}{bc}=\frac{8R^2+2Rr-p^2}{2Rrp^2};\quad \sum\frac{\operatorname{ctg}^2\frac{A}{2}}{bc}=\frac{2R-r}{2Rr^2};$$

$$\sum\frac{\sec^2\frac{A}{2}}{bc}=\frac{4R+r}{rp^2};\quad \sum\frac{\operatorname{cosec}^2\frac{A}{2}}{bc}=\frac{1}{r^2};\quad \sum r_a\operatorname{ctg}^2\frac{A}{2}=\frac{p^2}{r};\quad \sum\frac{\operatorname{tg}^2\frac{A}{2}}{r_a}=\frac{4R+r}{p^2};$$

$$\sum r_a\sin^2\frac{A}{2}=\frac{8R^2+2Rr-p^2}{2R};\quad \sum\frac{\sin^2\frac{A}{2}}{r_a}=\frac{1}{2R};\quad \sum r_a\cos^2\frac{A}{2}=\frac{p^2}{2R};$$

$$\sum\frac{\operatorname{tg}^2\frac{A}{2}}{h_a}=\frac{8R^2+2Rr-p^2}{rp^2};\quad \sum h_a\operatorname{ctg}^2\frac{A}{2}=\frac{p^2}{2Rr^2}\left[p^2+r^2-12Rr\right];$$

$$\sum\frac{\cos^2\frac{A}{2}}{r_a}=\frac{2R-r}{2Rr};\quad \sum h_a\sin^2\frac{A}{2}=\frac{r}{2R}\cdot(4R+r);\quad \sum\frac{\sin^2\frac{A}{2}}{h_a}=\frac{R-r}{2Rr};$$

$$\sum h_a\cos^2\frac{A}{2}=\frac{p^2}{2R};\quad \sum\frac{\cos^2\frac{A}{2}}{h_a}=\frac{R+r}{2Rr};\quad \sum r_a\operatorname{cosec}^2\frac{A}{2}=\frac{p^2+r^2+4Rr}{r};$$

$$\sum\frac{\operatorname{cosec}^2\frac{A}{2}}{h_a}=\frac{2R}{r^2};\quad \sum\frac{\operatorname{cosec}^2\frac{A}{2}}{r_a}=\frac{p^2+r^2-12Rr}{r^3};\quad \sum\frac{\sec^2\frac{A}{2}}{h_a}=\frac{2R(4R+r)}{rp^2};$$

$$\sum\frac{1}{1-\operatorname{tg}\frac{B}{2}\operatorname{tg}\frac{C}{2}}=\frac{p^2+r^2+4Rr}{4Rr}\geq\frac{9}{2};\quad \sum\frac{1}{\operatorname{ctg}\frac{B}{2}\operatorname{ctg}\frac{C}{2}-1}=\frac{p^2+r^2-8Rr}{4Rr}\geq\frac{3}{2};$$

$$\sum\frac{\operatorname{ctg}^2\frac{A}{2}}{p-a}=\frac{p}{r^2};\quad \sum\frac{\operatorname{tg}^2\frac{A}{2}}{(p-b)(p-c)}=\frac{4R+r}{rp^2};\quad \sum\frac{\sin^2\frac{A}{2}}{(p-b)(p-c)}=\frac{1}{2Rr};$$

$$\sum\frac{\cos^2\frac{A}{2}}{p-a}=\frac{p}{2Rr};\quad 4R\sum\sin\frac{A}{2}+r\sum\frac{1}{\sin\frac{A}{2}}=p\sum\frac{1}{\cos\frac{A}{2}};$$

$$AI=\frac{r}{\sin\frac{A}{2}};\quad \sum aAI^2=\frac{abc}{p}\sum\cos\frac{A}{2};\quad \sum AI\cdot BI=\frac{abc}{p}\sum\sin\frac{A}{2};\quad I_bI_c=4R\cos\frac{A}{2}.$$

## 3. Inegalități uzuale

$\sum\operatorname{tg}A\geq\frac{p}{r}\geq 3r\geq 3\sqrt{3}$ (în triunghiul ascuțitunghic); $\sum\operatorname{tg}^2 A\geq 9$ (în triunghiul ascuțitunghic); $\prod\operatorname{tg}A\geq 3\sqrt{3}$ (în triunghiul ascuțitunghic);

$\sum\operatorname{tg}^2\frac{A}{2}\operatorname{tg}^2\frac{B}{2}\geq\frac{1}{3}$;

$\sum\operatorname{ctg}A\geq\frac{(R+r)^2}{S}\geq\sqrt{3}$; $\sum\operatorname{ctg}^2 A\geq 1$; $\sum\operatorname{ctg}^n A\geq 3\cdot 3^{\frac{-n}{2}}$, $n\in\mathbb{N}$;

$\prod\operatorname{ctg}A\leq\frac{\sqrt{3}}{9}$ (în triunghiul ascuțitunghic);

$\sum\frac{1}{\sin 2A}\geq\frac{9R^2}{2S}\geq\frac{p^2+(R+r)^2}{2S}\geq\frac{2(4R+r)}{p}\geq 2\sqrt{3}$;

$\sum\sin\frac{A}{2}\leq\frac{3}{2}$; $\sum\frac{1}{\sin\frac{A}{2}}\geq 6$; $\sum\cos\frac{A}{2}\leq\frac{3\sqrt{3}}{3}$; $\sum\frac{1}{\sin\frac{A}{2}\sin\frac{B}{2}}\geq 12$;

$\sum\sin\frac{A}{2}\sin\frac{B}{2}\leq\frac{3}{4}$; $\sum\cos\frac{A}{2}\cos\frac{B}{2}\leq\frac{9}{4}$; $\sum\frac{1}{\cos A}\geq 6$ (în triunghiul ascuțitunghic);

$\sum\sec A\geq 6$ (în triunghiul ascuțitunghic); $\sum\sec^2 A\geq 12$ (în triunghiul ascuțitunghic);

$\sum\sec A\sec B\geq 12$ (în triunghiul ascuțitunghic);

$\sum\sec^2\frac{A}{2}\geq 4$; $\sum\operatorname{cosec}A\geq 2\sqrt{3}$; $\sum\operatorname{cosec}^2 A\geq 4$; $\sum\operatorname{cosec}\frac{A}{2}\geq 6$;

$\sum\operatorname{cosec}^2\frac{A}{2}\geq 12$; $\sum\operatorname{cosec}\frac{A}{2}\operatorname{cosec}\frac{B}{2}\geq 12$;

$\frac{\sqrt{3}}{R}\leq\sum\frac{1}{a}\leq\frac{p}{3Rr}\leq\frac{\sqrt{3}}{2r}$ (Petrovič).

# capitolul 1

# Inegalități cu unghiuri. Inegalitatea lui Jensen

*„Ai mii de motive să eșuezi în viață, dar nici măcar o scuză."*

Rudyard Kipling (1865–1936)

Să se arate că în orice triunghi au loc inegalitățile:

**1.1.** a) $0 < \sin A + \sin B + \sin C \leq \dfrac{3\sqrt{3}}{2}$, în orice triunghi.

*Period. Math. 1925, A. Padoa*

b) $2 < \sin A + \sin B + \sin C \leq \dfrac{3\sqrt{3}}{2}$, în triunghi ascuțitunghic.

c) $0 < \sin A + \sin B + \sin C < 1 + \sqrt{2}$, în triunghi obtuzunghic.

*O. Bottema, Euclid 1954/55*

**1.2.** a) $0 < \sin^2 A + \sin^2 B + \sin^2 C \leq \dfrac{9}{4}$, în orice triunghi.

b) $2 < \sin^2 A + \sin^2 B + \sin^2 C \leq \dfrac{9}{4}$, în triunghi ascuțitunghic.

c) $0 < \sin^2 A + \sin^2 B + \sin^2 C < 2$, în triunghi obtuzunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.3.** a) $\sin A + \sin B + \sin C \geq \sin 2A + \sin 2B + \sin 2C$. *O. Bottema, cdp*

b) Arătați că $\sin A + \sin B + \sin C = \sin 2A + \sin 2B + \sin 2C$ dacă și numai dacă $\Delta ABC$ este echilateral.

*GM 3/2007, Vasile Berghea, Avrig, Sibiu*

c) $\sin 2A + \sin 2B + \sin 2C = \dfrac{2rp}{R^2}$.

d) $\sin 2A + \sin 2B + \sin 2C \leq \dfrac{3\sqrt{3}}{2}$. *GM 12/2004, ****

e) $\sin 2A \sin 2B \sin 2C \leq \dfrac{p}{4R}$. *GM 4/2015, Florin Rotaru, Focșani*

**1.4.** a) $\sqrt{\sin A} + \sqrt{\sin B} + \sqrt{\sin C} \leq 3\sqrt[4]{\dfrac{3}{4}}$. *GMB 1963, Toma Albu, București*

b) $\sqrt[n]{\sin A}+\sqrt[n]{\sin B}+\sqrt[n]{\sin C}\le 3\sqrt[n]{\frac{\sqrt{3}}{2}}$, unde $n\in\mathbb{N}$, $n\ge 2$.

*GMB 2/1965, Valeriu Leibovici, elev, București*

c) $(\sin A)^{\alpha}+(\sin B)^{\alpha}+(\sin C)^{\alpha}\le 3(\sin 60^{\circ})^{\alpha}$, unde $0\le\alpha\le 1$.

d) $\dfrac{\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C}}{\sqrt{\operatorname{ctg}\frac{A}{2}}+\sqrt{\operatorname{ctg}\frac{B}{2}}\sqrt{\operatorname{ctg}\frac{C}{2}}}\ge\sqrt{\dfrac{r}{R}}$. *GM 3/2003, Marian Ursărescu, Roman*

**1.5.** a) $0<\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$, în orice triunghi. *A.M.M. 1937, K.P. Wilkins*

b) $0<\sin A\sin B\sin C\le\frac{3\sqrt{3}}{8}$, în triunghi ascuțitunghic.

c) $0<\sin A\sin B\sin C\le\frac{\sqrt{3}}{3}$, în triunghi obtuzunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.6.** $1<\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\le\frac{3}{2}$. *Math. Gazette 1939, J.M. Child*

**1.7.** $0<\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\le\frac{1}{8}$. *1897, T. Radó, A.M.M. 1932*

**1.8.** $\frac{3}{4}\le\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}<1$.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.9.** $1<\cos A+\cos B+\cos C\le\frac{3}{2}$. *Math. Gazette 1939, J.M. Child*

**1.10.** a) $\frac{3}{4}\le\cos^2 A+\cos^2 B+\cos^2 C<3$, în orice triunghi.

b) $\frac{3}{4}\le\cos^2 A+\cos^2 B+\cos^2 C<1$, în triunghi ascuțitunghic.

c) $1<\cos^2 A+\cos^2 B+\cos^2 C<3$, în triunghi obtuzunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.11.** $\cos A\cos B+\cos B\cos C+\cos C\cos A\le\frac{3}{4}$. *Math. Gazette 1939, J.M. Child*

**1.12.** a) $\cos A\cos B\cos C\le\frac{1}{8}$. *GMB 1925, C.C. Popovici*

b) $-1<\cos A\cos B\cos C\le\frac{1}{8}$, în orice triunghi.

c) $0\le\cos A\cos B\cos C\le\frac{1}{8}$, în triunghi neobtuzunghic.

d) $-1<\cos A\cos B\cos C<0$, în triunghi obtuzunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.13.** a) $\cos^2(B-C) \geq 8\cos A\cos B\cos C$. b) $\prod \cos A \leq \frac{1}{24}\sum \cos^2(B-C)$.

*C. Coşniţă şi F. Turtoiu, 1965, Bucureşti, cdp*

c) În $\Delta ABC$ neobtuzunghic: $\prod \cos(B-C) \geq 8\prod \cos A$ şi

$$\prod \cos(B-C) \geq 64\prod \cos^2 A.$$

**1.14.** $2 < \cos\frac{A}{2} + \cos\frac{B}{2} + \cos\frac{C}{2} \leq \frac{3\sqrt{3}}{2}$.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.15.** a) $0 < \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{8}$, în orice triunghi.

b) $\frac{1}{2} < \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \leq \frac{3\sqrt{3}}{8}$, în triunghi ascuţitunghic.

c) $0 < \cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} < \frac{3\sqrt{3}}{8}$, în triunghi obtuzunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.16.** $2 < \cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2} \leq \frac{9}{4}$.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.17.** a) $\mathrm{tg}^2 A + \mathrm{tg}^2 B + \mathrm{tg}^2 C > 0$, în orice triunghi.

b) $\mathrm{tg}^2 A + \mathrm{tg}^2 B + \mathrm{tg}^2 C \geq 9$, în triunghi ascuţitunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.18.** a) $\mathrm{tg} A\,\mathrm{tg} B\,\mathrm{tg} C \geq 3\sqrt{3}$, în triunghi ascuţitunghic.

b) $\mathrm{tg} A\,\mathrm{tg} B\,\mathrm{tg} C < 0$, în triunghi obtuzunghic.

*R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.19.** a) $\mathrm{tg}\frac{A}{2} + \mathrm{tg}\frac{B}{2} + \mathrm{tg}\frac{C}{2} \geq \sqrt{3}$. *J. Karamatra, GMFA 1948*

b) $\mathrm{tg}^2\frac{A}{2} + \mathrm{tg}^2\frac{B}{2} + \mathrm{tg}^2\frac{C}{2} \geq 1$.

*C.V. Durell şi A. Robson, Advanced Trigonometry, Londra, 1948*

c) $\mathrm{tg}^6\frac{A}{2} + \mathrm{tg}^6\frac{B}{2} + \mathrm{tg}^6\frac{C}{2} \geq \frac{1}{9}$. d) $\mathrm{tg}^n\frac{A}{2} + \mathrm{tg}^n\frac{B}{2} + \mathrm{tg}^n\frac{C}{2} \geq \frac{3}{\sqrt{3^n}}$, unde $n \in \mathbb{N}$.

**1.20.** $0 < \mathrm{tg}\frac{A}{2}\mathrm{tg}\frac{B}{2}\mathrm{tg}\frac{C}{2} \leq \frac{\sqrt{3}}{9}$. *R. Kooistra, Nieuw Tijdschr. Wisk. 45, 1957/58*

**1.21.** $\sum\sqrt{5 + \mathrm{tg}\frac{A}{2}\mathrm{tg}\frac{B}{2}} \leq 4\sqrt{3}$. *Matematica v škole 1953, J.A. Izosimov*

**1.22.** $\mathrm{ctg} A + \mathrm{ctg} B + \mathrm{ctg} C \geq \sqrt{3}$. *Bull. Soc. Math., Grecia, 1934, T. Varopoulos*